



8.9 The composition of linear maps

Douglas Wilhelm Harder, LEL, M.Math.

dwharder@uwaterloo.ca

dwharder@gmail.com





Introduction

- In this topic, we will
 - Define composition of linear maps
 - Examine some of the properties
 - Learn that composition is not commutative
 - Determine how to calculate the composition of matrices
 - That is, matrix-matrix multiplication
 - Learn under which circumstances we can multiply two matrices





Composition of linear maps

- Given one linear map $A: \mathcal{U} \rightarrow \mathcal{V}$ and a second linear map $B: \mathcal{V} \rightarrow \mathcal{W}$, then we can define the composition $BA: \mathcal{U} \rightarrow \mathcal{W}$ where
$$(BA)\mathbf{u} = B(A\mathbf{u}).$$

Theorem

The composition of linear maps is a linear map.

Proof:

$$\begin{aligned}(BA)(\alpha \mathbf{u}_1 + \beta \mathbf{u}_2) &= B(A(\alpha \mathbf{u}_1 + \beta \mathbf{u}_2)) \\ &= B(\alpha A\mathbf{u}_1 + \beta A\mathbf{u}_2) \\ &= \alpha B(A\mathbf{u}_1) + \beta B(A\mathbf{u}_2) \\ &= \alpha (BA)\mathbf{u}_1 + \beta (BA)\mathbf{u}_2 \quad \blacksquare\end{aligned}$$





Composition is associative

Theorem

Given three linear maps $A: \mathcal{U} \rightarrow \mathcal{V}$, $B: \mathcal{V} \rightarrow \mathcal{W}$ and $C: \mathcal{W} \rightarrow \mathcal{X}$, then $(CB)A = C(BA)$.

- That is, composition is associative

Proof:

$$\begin{aligned} ((CB)A) \mathbf{u} &= (CB)(A\mathbf{u}) \\ &= C(B(A\mathbf{u})) \\ &= C((BA)\mathbf{u}) \\ &= (C(BA)) \mathbf{u} \quad \blacksquare \end{aligned}$$





Composition distributes over map addition

Theorem

Given three linear maps $A_1, A_2: \mathcal{U} \rightarrow \mathcal{V}$ and $B: \mathcal{V} \rightarrow \mathcal{W}$, then $B(A_1 + A_2) = BA_1 + BA_2$.

– That is, composition distributes over map addition

Proof:

$$\begin{aligned}(B(A_1 + A_2)) \mathbf{u} &= B((A_1 + A_2)\mathbf{u}) \\ &= B(A_1\mathbf{u} + A_2\mathbf{u}) \\ &= B(A_1\mathbf{u}) + B(A_2\mathbf{u}) \\ &= (BA_1) \mathbf{u} + (BA_2) \mathbf{u} \\ &= (BA_1 + BA_2) \mathbf{u} \blacksquare\end{aligned}$$





Composition distributes over map addition

Theorem

Given three linear maps $A: \mathcal{U} \rightarrow \mathcal{V}$ and $B_1, B_2: \mathcal{V} \rightarrow \mathcal{W}$, then $(B_1 + B_2)A = B_1A + B_2A$.

– That is, composition distributes over map addition

Proof:

$$\begin{aligned} ((B_1 + B_2)A) \mathbf{u} &= (B_1 + B_2)(A\mathbf{u}) \\ &= B_1(A\mathbf{u}) + B_2(A\mathbf{u}) \\ &= (B_1A)\mathbf{u} + (B_2A)\mathbf{u} \\ &= (B_1A + B_2A) \mathbf{u} \blacksquare \end{aligned}$$





Composition is bilinear

Theorem

Given linear maps $A: \mathcal{U} \rightarrow \mathcal{V}$ and $B: \mathcal{V} \rightarrow \mathcal{W}$ and a scalar α , then $\alpha(BA) = (\alpha B)A = B(\alpha A)$.

– That is, composition is bilinear

Proof:

$$\begin{aligned} (\alpha(BA)) \mathbf{u} &= \alpha ((BA)\mathbf{u}) \\ &= \alpha B(A\mathbf{u}) &= B(\alpha(A\mathbf{u})) \\ &= (\alpha B)(A\mathbf{u}) &= B((\alpha A)\mathbf{u}) \\ &= ((\alpha B) A)\mathbf{u} &= (B(\alpha A)) \mathbf{u} \blacksquare \end{aligned}$$





Composition is not commutative

- First, if $A: \mathcal{U} \rightarrow \mathcal{V}$ and $B: \mathcal{V} \rightarrow \mathcal{W}$, then AB is not even defined
- Second, if $A: \mathcal{U} \rightarrow \mathcal{V}$ and $B: \mathcal{V} \rightarrow \mathcal{U}$,
then $BA: \mathcal{U} \rightarrow \mathcal{U}$ while $AB: \mathcal{V} \rightarrow \mathcal{V}$
- Thus, we can only even begin to consider commutativity if
 $A, B: \mathcal{U} \rightarrow \mathcal{U}$
- To show that composition of linear maps is not commutative,
we only must find one pair of linear maps between any two
vector spaces we want that do not commute

$$\text{proj}_{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \left(\text{proj}_{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \text{proj}_{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\text{proj}_{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \left(\text{proj}_{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \text{proj}_{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$$





What matrix is the composition of matrices?

- Given a linear map $A:\mathbf{R}^n \rightarrow \mathbf{R}^m$ and $B:\mathbf{R}^m \rightarrow \mathbf{R}^\ell$,
then we know that $BA:\mathbf{R}^n \rightarrow \mathbf{R}^\ell$ is a linear map
- We know that all linear maps between finite-dimensional vector spaces representable by matrices, so
 - If A is an $m \times n$ matrix and B is an $\ell \times m$ matrix,
it follows that BA is an $\ell \times n$ matrix, but what matrix?





What matrix is the composition of matrices?

- Recall that if $A = (\mathbf{a}_1 \cdots \mathbf{a}_n)$ then $A\mathbf{u} = u_1\mathbf{a}_1 + \cdots + u_n\mathbf{a}_n$, so

$$\begin{aligned}(BA)\mathbf{u} &= B(A\mathbf{u}) = B(u_1\mathbf{a}_1 + \cdots + u_n\mathbf{a}_n) \\ &= u_1B\mathbf{a}_1 + \cdots + u_nB\mathbf{a}_n \\ &= (B\mathbf{a}_1 \cdots B\mathbf{a}_n)\mathbf{u}\end{aligned}$$

- Thus, if $A = (\mathbf{a}_1 \cdots \mathbf{a}_n)$ then $BA = (B\mathbf{a}_1 \cdots B\mathbf{a}_n)$,
that is, multiply each column vector of A by B





What matrix is the composition of matrices?

- For example, consider these two matrices:

$$A = \begin{pmatrix} 2 & 1 & -3 \\ 0 & 4 & -2 \end{pmatrix} \quad B = \begin{pmatrix} -1 & 3 \\ 2 & 0 \\ 4 & 1 \\ 5 & -3 \end{pmatrix}$$

- We see $A: \mathbf{R}^3 \rightarrow \mathbf{R}^2$ and $B: \mathbf{R}^2 \rightarrow \mathbf{R}^4$, so $BA: \mathbf{R}^3 \rightarrow \mathbf{R}^4$ and

$$BA = \left(\begin{pmatrix} -1 & 3 \\ 2 & 0 \\ 4 & 1 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad \begin{pmatrix} -1 & 3 \\ 2 & 0 \\ 4 & 1 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \end{pmatrix} \quad \begin{pmatrix} -1 & 3 \\ 2 & 0 \\ 4 & 1 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} -3 \\ -2 \end{pmatrix} \right) = \begin{pmatrix} -2 & 11 & -3 \\ 4 & 2 & -6 \\ 8 & 8 & -14 \\ 10 & -7 & -9 \end{pmatrix}$$





What matrix is the composition of matrices?

- There is a more algorithmic approach to finding the composition, or matrix-matrix product:

$$BA = \begin{pmatrix} -1 & 3 \\ 2 & 0 \\ 4 & 1 \\ 5 & -3 \end{pmatrix}_{4 \times 2} \begin{pmatrix} 2 & 1 & -3 \\ 0 & 4 & -2 \end{pmatrix}_{2 \times 3} = \begin{pmatrix} -2 & 11 & -3 \\ 4 & 2 & -6 \\ 8 & 8 & -14 \\ 10 & -7 & -9 \end{pmatrix}_{4 \times 3}$$





Which matrices can be multiplied?

- Important: If the dimensions of the matrices do not align, it makes no sense to discuss matrix-matrix multiplication
 - You cannot multiply a 5-dimensional vector by a 5×4 matrix
 - A 5×4 matrix maps $\mathbf{R}^4$ to $\mathbf{R}^5$
 - Similarly, you cannot multiply a 4×7 matrix by a 3×4 matrix, because the one on the right maps $\mathbf{R}^4$ to $\mathbf{R}^3$,
but the second maps a 7-dimensional vector to $\mathbf{R}^4$
 - However, you can multiply a 3×4 matrix by a 4×7 matrix

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}_{3 \times 4} \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}_{4 \times 7} \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}_{3 \times 4} \equiv \begin{pmatrix} * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \end{pmatrix}_{3 \times 7}$$





Summary

- Following this topic, you now
 - Know the definition of the composition of linear maps
 - Understand it to be associative, mapping over addition of linear maps, and is bilinear
 - Know that it is not commutative
 - Know how to find the composition of matrices:
 - Matrix-matrix multiplication
 - Understand how to determine which matrices can be multiplied





References

- [1] https://en.wikipedia.org/wiki/Linear_map#Forming_new_linear_maps_from_given_ones
- [2] https://en.wikipedia.org/wiki/Matrix_multiplication





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas. Mathematical equations are prepared in MathType by Design Science, Inc. Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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